Shift Invariance, Incomplete Arrays and Coupled CPD: a Case Study

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Abstract—**Tensors have proven to be useful tools for array processing. Most attention has been paid to separable arrays, which lead to a Canonical Polyadic Decomposition (CPD). For more general geometries, and in particular for sparse arrays and arrays with missing sensors, more general tensor methods are required. The recently proposed coupled CPD framework allows a data fission/fusion approach in which one zooms in on partial structures and combines the partial CPDs through which the latter are imposed. This approach yields explicit algebraic conditions under which the solution is unique. The exact solution can be found with a matrix eigenvalue decomposition in the noiseless case, similar to ESPRIT in the case of uniform linear arrays. We study in in detail the case of sparse spatial sampling where sensors are located on points of a two-dimensional grid. Despite the fact that the array is incomplete, coupled CPD allows us to exploit the rectangularity of the grid as well as the uniformity of the spatial sampling in both dimensions.**

I. Introduction

In the past two decades it has become clear that several problems in array processing can be solved using tensor decompositions. However, most of the existing results (e.g., [11], [22], [12]) are limited to basic array processing problems that can be formulated by means of the Canonical Polyadic Decomposition (CPD) [6], [10], [9], [1], [2]. The authors have in [18], [19], [15], [16] explained that much more general array processing problems can be solved by means of coupled CPD [17], [20]. This includes multidimensional Harmonic Retrieval (HR) problems [18], sparse array processing problems [15] and array processing problems involving multiresolution/multirate sampling structures [19], [16]. The general strategy is a combination of data fission and data fusion. From the global, difficult problem we first derive a set of subproblems, each of which can be solved by means of a simple CPD. The resulting CPDs are next combined by taking into account the coupling between factor matrices. This approach leads to explicit algebraic uniqueness conditions and an algorithm based on Generalized EigenValue Decomposition (GEVD) that is guaranteed to find the solution in the noiseless case. The estimates may be refined by optimization [13], [14] if desired. In other words, the coupled CPD approach allows one to generalize ESPRIT [11] for the basic uniform linear array to much more general problems. The present paper is meant to help the reader get acquainted with approach. It details as a case study how to proceed for incomplete Uniform Rectangular Arrays (URAs) (e.g., thinned URA [7]). The incomplete URA example does not appear as such in [18], [19], [16], [15]. We will eventually show simulations results for the incomplete URA in Figure 1 (Left).

Sections II and III briefly review tensor-based array processing and the coupled CPD of tensors with missing fibers, respectively. Section IV presents a link between non-separable arrays that enjoy shift-invariance and/or Khatri-Rao structures and the coupled CPD of tensors that have fibers missing. We conclude in Section V.

Notation: Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively. The symbols $\otimes$ and $\odot$ denote the Kronecker and Khatri-Rao product,

$$\mathbf{A}\otimes\mathbf{B} := \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}, \ \mathbf{A}\odot\mathbf{B} := [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \mathbf{a}_2 \otimes \mathbf{b}_2 \ \dots],$$

in which $(\mathbf{A})_{mn} = a_{mn}$. The symbol $*$ denotes the Hadamard product, e.g. $(\mathcal{A} * \mathcal{B})_{ijk} = a_{ijk}b_{ijk}$ in the case of third-order tensors. The outer product of three vectors $\mathbf{a} \in \mathbb{C}^I$, $\mathbf{b} \in \mathbb{C}^J$ and $\mathbf{c} \in \mathbb{C}^K$ is denoted by $\mathbf{a} \circ \mathbf{b} \circ \mathbf{c} \in \mathbb{C}^{I\times J\times K}$, such that $(\mathbf{a} \circ \mathbf{b} \circ \mathbf{c})_{ijk} = a_i b_j c_k$. The binomial coefficient is denoted by $C_m^k = \frac{m!}{k!(m-k)!}$. Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, then $C_2(\mathbf{A}) \in \mathbb{C}^{C_m^2 \times C_n^2}$ denotes the compound matrix containing the determinants of all 2×2 submatrices of $\mathbf{A}$, arranged with the submatrix index sets in lexicographic order [8], [2]. The transpose and pseudoinverse of a matrix $\mathbf{A}$ are denoted by $\mathbf{A}^T$ and $\mathbf{A}^\dagger$, respectively. $\mathbf{I}_N \in \mathbb{C}^{N\times N}$ denotes the identity matrix. The number of non-zero entries of a vector $\mathbf{a}$ is denoted by $\omega(\mathbf{a})$. $\text{Diag}(\mathbf{a}) \in \mathbb{C}^{I\times I}$ denotes the diagonal matrix holding the column vector $\mathbf{a} \in \mathbb{C}^I$ on its diagonal. Given $\mathbf{A} \in \mathbb{C}^{I\times J}$, $\text{Vec}(\mathbf{A}) \in \mathbb{C}^{IJ}$ denotes the column vector $\text{Vec}(\mathbf{A}) = [a_{1,1}, a_{1,2}, \dots, a_{I,J}]^T$. Denoting the submatrix of $\mathbf{A} \in \mathbb{C}^{I\times R}$ consisting of rows from k to l by $\mathbf{A}(k{:}l,:)$ we also write $\downarrow_m (\mathbf{A}) = \mathbf{A}(m+1 : I, :)$ and $\uparrow^m (\mathbf{A}) = \mathbf{A}(1 : I - m, :)$.

II. Tensor-based array processing

It is well-known that several Direction-Of-Arrival (DOA) estimation problems in array processing can be cast into tensors $\mathcal{X} \in \mathbb{C}^{I\times J\times K}$ admitting a (constrained) CPD given by [12]:

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{s}_r\,, \tag{1}$$

where the columns of the factor matrices $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R] \in \mathbb{C}^{I\times R}$ and $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R] \in \mathbb{C}^{J\times R}$ are subject to constraints depending on the given antenna array configuration. The data snapshot matrix $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R] \in \mathbb{C}^{K\times R}$ holds the R impinging signals of length K on its columns. In the case of URAs, the columns of $\mathbf{a}_r$ and $\mathbf{b}_r$ in (1) are Vandermonde (e.g., [21], [3], [12], [23], [5], [4], [18]):

$$\mathbf{a}_r = [1, x_r, x_r^2, \ldots, x_r^{I-1}]^T, \quad \mathbf{b}_r = [1, y_r, y_r^2, \ldots, y_r^{J-1}]^T. \tag{2}$$

From (2) it is clear that a URA factorization problem can also be interpreted as a two-dimensional HR problem. Since $\downarrow_m(\mathbf{A}) = \uparrow^m(\mathbf{A})\cdot\mathrm{Diag}\left([x_1^m, \ldots, x_R^m]^T\right)$, the Vandermonde matrix $\mathbf{A}$ is said to be shift-invariant. (Similarly for $\mathbf{B}$.) This shift-invariance structure can be used to transform a two-dimensional HR problem into a coupled CPD problem [18]. In this paper we consider the more complicated case of an incomplete URA.

In DOA estimation, the goal is to find the generators $\{x_r, y_r\}$ in (2) from the observed data tensor $\mathcal{X}$. A notable limitation of the CPD-based approach is that it only supports separable arrays in which the observation tensor $\mathcal{X}$ must admit a factorization of the form (1), i.e., arrays that can be constructed from an outer product of one-dimensional arrays. An extension to some nonseparable arrays (e.g., L-shaped) via the coupled CPD model [17], [20] can be found in [15]. Briefly, the idea is to consider the nonseparable array as a combination of separable subarrays, express the CPDs of the latter and observe that they are coupled via the matrix $\mathbf{S}$. In this paper we illustrate the approach for quite an irregular configuration obtained by sparse spatial sampling.

Incomplete URA: In this paper we consider antenna arrays in which the sensors are located on an $(I \times J)$ two-dimensional grid such that the output of the sensor indexed by the pair (i, j) at the kth time snapshot corresponds to the (i, j, k) entry of the tensor $\mathcal{X}$ in (1). We say that the tensor $\mathcal{X}$ in (3) is missing a fiber if for some pair $(i, j) \in \{1, \ldots, I\}\times\{1, \ldots, J\}$ the vector $\mathbf{x}_{ij\bullet} \in \mathbb{C}^K$, defined by $(\mathbf{x}_{ij\bullet})_k = x_{ijk}$, is unobserved.

From relation (1) it is clear that the URA observation tensor with missing fibers can be written as

$$\mathcal{Y} = \mathcal{W} * \mathcal{X} = \mathcal{W} * \Big(\sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{s}_r\Big) \in \mathbb{C}^{I\times J\times K}, \tag{3}$$

where the fibers of the binary indicator tensor $\mathcal{W} \in \{0, 1\}^{I\times J\times K}$ are given by

$$\mathbf{w}_{ij\bullet} = \begin{cases} \mathbf{1}_K, \text{ if fiber } \mathbf{x}_{ij\bullet} \text{ is observed,} \\ \mathbf{0}_K, \text{ otherwise,} \end{cases}$$

where $\mathbf{1}_K \in \mathbb{C}^K$ and $\mathbf{0}_K \in \mathbb{C}^K$ denote the all-ones and all-zeros vector, respectively. Summarizing, we consider the incomplete URA as a complete URA that yields a data tensor of which a number of fibers are not observed. This point of view will allow us to impose shift invariance in the case of an incomplete array.

III. Coupled CPD of tensors that have fibers missing

We say that a collection of tensors $\mathcal{X}^{(m)} \in \mathbb{C}^{I_m\times J_m\times K}$, $m \in \{1, \ldots, M\}$, admits an R-term coupled polyadic decomposition if each tensor $\mathcal{X}^{(m)}$ can be written as [17]:

$$\mathcal{X}^{(m)} = \sum_{r=1}^{R} \mathbf{a}_r^{(m)} \otimes \mathbf{b}_r^{(m)} \otimes \mathbf{s}_r, \quad m \in \{1, \ldots, M\}, \tag{4}$$

with factor matrices $\mathbf{A}^{(m)} = [\mathbf{a}_1^{(m)}, \ldots, \mathbf{a}_R^{(m)}] \in \mathbb{C}^{I_m\times R}$, $\mathbf{B}^{(m)} = [\mathbf{b}_1^{(m)}, \ldots, \mathbf{b}_R^{(m)}] \in \mathbb{C}^{J_m\times R}$ and $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_R] \in \mathbb{C}^{K\times R}$. We define the coupled rank of $\{\mathcal{X}^{(m)}\}$ as the minimal number of coupled rank-1 tensors $\mathbf{a}_r^{(m)} \otimes \mathbf{b}_r^{(m)} \otimes \mathbf{s}_r$ that yield $\{\mathcal{X}^{(m)}\}$ in a linear combination. If the coupled rank of $\{\mathcal{X}^{(m)}\}$ is R, then (4) is called the coupled CPD of $\{\mathcal{X}^{(m)}\}$.

The coupled CPD of a collection of tensors $\{\mathcal{X}^{(m)}\}$ that have missing fibers is denoted by:

$$\mathcal{Y}^{(m)} = \mathcal{W}^{(m)} * \Big(\sum_{r=1}^{R} \mathbf{a}_r^{(m)} \otimes \mathbf{b}_r^{(m)} \otimes \mathbf{s}_r\Big), \quad m \in \{1, \ldots, M\}, \tag{5}$$

where $w_{ijk}^{(m)} = 0$, $\forall k \in \{1, \ldots, K\}$, if fiber $\mathbf{x}_{ij\bullet}^{(m)}$ is missing and $w_{ijk}^{(m)} = 1$ otherwise. Under certain conditions, (5) can be expressed as the coupled CPD of a set of fully observed tensors. This encompasses the case of a single incomplete tensor with missing fibers. In a nutshell, the approach consists of stacking all the $(2 \times 2 \times K)$ subtensors that are fully observed, and coupling the decomposition of the latter. Explicit sufficient uniqueness conditions for the coupled CPD of tensors with missing fibers can then be obtained.

In this paper we will mainly be interested in the recovery of the common factor $\mathbf{S}$ in (4) from the partially observed tensors $\{\mathcal{Y}^{(m)}\}$. We say that the common factor matrix $\mathbf{S}$ is essentially unique when it is only subject to column scaling and permutation ambiguities. Proposition 3.1 below provides a sufficient condition for essential uniqueness of the common factor matrix $\mathbf{S}$. It will make use of a binary diagonal matrix $\mathbf{D}_{\text{sel}}^{(m)} \in \{0, 1\}^{C_{I_m}^2 C_{J_m}^2\times C_{I_m}^2 C_{J_m}^2}$ that holds the vector $\mathbf{d}_{\text{sel}}^{(m)} \in \{0, 1\}^{C_{I_m}^2 C_{J_m}^2}$ on its diagonal, i.e., $\mathbf{D}_{\text{sel}}^{(m)} = \mathrm{Diag}\left(\mathbf{d}_{\text{sel}}^{(m)}\right)$. The entries of the vector

$$\mathbf{d}_{\text{sel}}^{(m)} = [w_{(1,2),(1,2)}^{(m)},\ w_{(1,2),(1,3)}^{(m)},\ \ldots,\ w_{(I_m-1,I_m),(J_m-1,J_m)}^{(m)}]^T \tag{6}$$

are given by

$$w_{(p,q),(u,v)}^{(m)} = \begin{cases} 1, \text{ if fibers } \mathbf{x}_{pu\bullet}^{(m)}, \mathbf{x}_{qu\bullet}^{(m)}, \mathbf{x}_{pv\bullet}^{(m)} \text{ and } \mathbf{x}_{qv\bullet}^{(m)} \text{ are observed,} \\ 0, \text{ otherwise.} \end{cases}$$

Proposition 3.1 will also make use of the matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{D}_{\text{sel}}^{(1)}\left(C_2(\mathbf{A}^{(1)}) \odot C_2(\mathbf{B}^{(1)})\right) \\ \vdots \\ \mathbf{D}_{\text{sel}}^{(M)}\left(C_2(\mathbf{A}^{(M)}) \odot C_2(\mathbf{B}^{(M)})\right) \end{bmatrix} \in \mathbb{C}^{\left(\sum_{m=1}^{M} C_{I_m}^2 C_{J_m}^2\right)\times C_R^2}. \tag{7}$$

Proposition 3.1: Consider the tensors $\mathcal{X}^{(m)} \in \mathbb{C}^{I_m\times J_m\times K}$, $m \in \{1, \ldots, M\}$, partially observed as $\mathcal{Y}^{(m)} = \mathcal{W}^{(m)} * \mathcal{X}^{(m)}$, and its coupled PD given by (4). If

$$\begin{cases} \mathbf{S} \text{ in (4) has full column rank,} \\ \mathbf{G} \text{ in (7) has full column rank,} \end{cases} \tag{8}$$

then the coupled rank of $\{\mathcal{X}^{(m)}\}$ is R and the factor matrix $\mathbf{S}$ is essentially unique. Generically, condition (8) is satisfied if $C_R^2 \leq \sum_{m=1}^{M} \omega(\mathbf{d}_{\text{sel}}^{(m)})$ and $R \leq K$.

In analogy with [1], [20], it can be shown that under condition (8), $\mathbf{S}$ can explicitly be obtained from a GEVD in the noiseless case.

IV. Incomplete URAs and coupled CPD

In this section we connect DOA estimation using incomplete URAs with the coupled CPD.

A. From incomplete URA to coupled CPD

The incomplete URA observation tensor $\mathcal{Y}$ in (3) can be seen as a collection of K matrices $\mathbf{Y}_1 := \mathcal{Y}(:,:,1),\ldots,\mathbf{Y}_K := \mathcal{Y}(:,:,K)$, each admitting the factorization

$$\mathbf{Y}_k = \mathbf{W} * (\mathbf{A}\ \text{Diag}(\tilde{\mathbf{s}}_k)\,\mathbf{B}^T),$$

where $\tilde{\mathbf{s}}_k = \text{Vec}(\mathbf{S}(k,:)) \in \mathbb{C}^R$ and where $\mathbf{W} := \mathcal{W}(:,:,1) = \cdots = \mathcal{W}(:,:,K) \in \{0,1\}^{I\times J}$ is a binary indicator matrix with property $w_{ij} = 1$ if fiber $\mathbf{x}_{ij\bullet}$ of the URA data tensor $\mathcal{X}$ in (3) is observed and zero otherwise. Vectorization and stacking yields

$$\mathbf{Y} := [\text{Vec}(\mathbf{Y}_1),\ldots,\text{Vec}(\mathbf{Y}_K)] = \mathbf{D}_{\mathbf{w}}(\mathbf{B}\odot\mathbf{A})\mathbf{S}^T \in \mathbb{C}^{IJ\times K}, \quad (9)$$

where $\mathbf{D}_{\mathbf{w}} = \text{Diag}\left(\text{Vec}\left(\mathbf{W}^T\right)\right)$. It will be explained in this section that the incomplete URA observation matrix $\mathbf{Y}$ in (9) involves three low-rank structures, namely the Khatri-Rao structure of $\mathbf{B}\odot\mathbf{A}$, the shift-invariance of $\mathbf{A}$ and the shift-invariance of $\mathbf{B}$. (It will become clear later why shift-invariance is denoted as a low-rank structure.) We first consider the three structures separately and combine them afterwards. More precisely, from tensor $\mathcal{Y}$ in (3) we derive a number of decompositions that each exploit only one structure and thereafter merge the results via a coupled CPD of tensors with missing fibers.

a) Exploiting the Khatri-Rao structure of $\mathbf{B}\odot\mathbf{A}$: Since (9) corresponds to a matrix representation of a CPD of a tensor with missing fibers, Proposition 3.1 with $M = 1$ can be applied. The matrix $\mathbf{G}$ in (7) is equal to

$$\mathbf{G}_{\mathbf{B}\odot\mathbf{A}} := \mathbf{D}_{\text{sel}}\left(C_2(\mathbf{A})\odot C_2(\mathbf{B})\right), \quad (10)$$

where the diagonal entries of $\mathbf{D}_{\text{sel}} = \text{Diag}(\mathbf{d}_{\text{sel}})$ are given by (6). Note that the superscripts of $\mathbf{d}_{\text{sel}}$ and $\mathbf{D}_{\text{sel}}$ have been dropped, i.e., $\mathbf{d}_{\text{sel}} := \mathbf{d}_{\text{sel}}^{(1)}$ and $\mathbf{D}_{\text{sel}} := \mathbf{D}_{\text{sel}}^{(1)}$.

b) Exploiting the shift-invariance structure of $\mathbf{A}$: The shift-invariance property $\downarrow_m(\mathbf{A}) = \uparrow^m(\mathbf{A})\cdot\text{Diag}\left([x_1^m,\ldots,x_R^m]^T\right)$ can be translated into a low-rank structure. Indeed, each column of $\begin{bmatrix}\uparrow^m(\mathbf{A})\\ \downarrow_m(\mathbf{A})\end{bmatrix} = \mathbf{A}^{(m)}\odot\uparrow^m(\mathbf{A})$ corresponds to a vectorized rank-one matrix, where $\mathbf{A}^{(m)} = \begin{bmatrix}1 & \cdots & 1\\ x_1^m & \cdots & x_R^m\end{bmatrix}$.

We will now build a two-slice tensor $\mathcal{Y}^{(m)} \in \mathbb{C}^{2\times(I-m)J\times K}$, the decomposition of which exploits this shift-invariance. First, denote the subtensors formed by the $I-m$ top and bottom horizontal slices of $\mathcal{Y}$ in (3) by $\uparrow_{(1)}^m \mathcal{Y} \in \mathbb{C}^{(I-m)\times J\times K}$ and $\downarrow_m^{(1)} \mathcal{Y} \in \mathbb{C}^{(I-m)\times J\times K}$, respectively. Matricization yields $\uparrow_{(1)}^m\mathbf{Y} \in \mathbb{C}^{(I-m)J\times K}$ and $\downarrow_m^{(1)}\mathbf{Y}\in\mathbb{C}^{(I-m)J\times K}$, which can be obtained from $\mathbf{Y}$ via $\uparrow_{(1)}^m\mathbf{Y}=(\mathbf{I}_J\otimes\uparrow^m(\mathbf{I}_I))\mathbf{Y}$ and $\downarrow_m^{(1)}\mathbf{Y}=(\mathbf{I}_J\otimes\downarrow_m(\mathbf{I}_I))\mathbf{Y}$. Substitution of (9) yields $\uparrow_{(1)}^m\mathbf{Y}=\uparrow_{(1)}^m\mathbf{W}*((\mathbf{B}\odot\uparrow^m(\mathbf{A})\mathbf{S}^T)$ and $\downarrow_m^{(1)}\mathbf{Y} =\downarrow_m^{(1)}\mathbf{W} * ((\mathbf{B}\odot\downarrow_m(\mathbf{A})\mathbf{S}^T)$, where $\uparrow_{(1)}^m\mathbf{W} = (\mathbf{I}_J\otimes\uparrow^m(\mathbf{I}_I))\mathbf{D}_{\mathbf{W}}$ and $\downarrow_m^{(1)}\mathbf{W} = (\mathbf{I}_J\otimes\downarrow_m(\mathbf{I}_I))\mathbf{D}_{\mathbf{W}}$. Stacking and making use of $\downarrow_m(\mathbf{A}) =\uparrow^m(\mathbf{A})\cdot\text{Diag}\left([x_1^m,\ldots,x_R^m]^T\right)$ yields

$$\begin{bmatrix}\uparrow_{(1)}^m\mathbf{Y}\\ \downarrow_m^{(1)}\mathbf{Y}\end{bmatrix} = \mathbf{D}_{\mathbf{W}}^{(m)}(\mathbf{A}^{(m)}\odot\mathbf{E}^{(m)})\mathbf{S}^T \in \mathbb{C}^{2(I-m)J\times K}, \quad (11)$$

where $\mathbf{D}_{\mathbf{W}}^{(m)} = \begin{bmatrix}\uparrow_{(1)}^m\mathbf{W} & \mathbf{0}\\ \mathbf{0} & \downarrow_m^{(1)}\mathbf{W}\end{bmatrix}$ is diagonal and $\mathbf{E}^{(m)} = \mathbf{B}\odot\uparrow^m(\mathbf{A})$. From (11) it is clear that by capitalizing on the shift-invariance property of $\mathbf{A}$, we obtain (cf. (3)):

$$\mathcal{Y}^{(m)} = \mathcal{W}_{\mathbf{A}}^{(m)} * \left(\sum_{r=1}^{R}\mathbf{a}_r^{(m)}\otimes\mathbf{e}_r^{(m)}\otimes\mathbf{s}_r\right) \in \mathbb{C}^{2\times(I-m)J\times K}, \quad (12)$$

with matrix slices $\mathcal{Y}^{(m)}(1,:,:)=\uparrow_{(1)}^m\mathbf{Y}$ and $\mathcal{Y}^{(m)}(2,:,:)=\downarrow_m^{(1)}\mathbf{Y}$. Similarly, the two matrix slices of $\mathcal{W}_{\mathbf{A}}^{(m)}$ are $\mathcal{W}_{\mathbf{A}}^{(m)}(1,:,:)=\uparrow_{(1)}^m\mathbf{W}$ and $\mathcal{W}_{\mathbf{A}}^{(m)}(2,:,:)=\downarrow_m^{(1)}\mathbf{W}$. We ignore the Khatri-Rao structure of $\mathbf{E}$ since the overall Khatri-Rao structure of $\mathbf{B}\odot\mathbf{A}$ has already been exploited in the construction of $\mathbf{G}_{\mathbf{B}\odot\mathbf{A}}$. In other words, we just see (12) as a CPD of a tensor with missing fibers of the form (3). Essential uniqueness conditions for $\mathbf{S}$ can now be derived from $\mathcal{Y}^{(m)}$. For a complete URA, $m = 1$ fully exploits the shift-invariance of $\mathbf{A}$ (i.e., we can use Proposition 3.1 with $M = 1$). In contrast, for an incomplete URA, several shift factors m in the interval $1 \leq m < I$ may have to be considered (i.e., we use Proposition 3.1 with $M > 1$). We obtain the coupled CPD of the set of tensors $\{\mathcal{Y}^{(m)}\}_{1\leq m<I}$ with missing fibers. [1] Matrix $\mathbf{G}$ in (7) is equal to[2]

$$\mathbf{G}_{\mathbf{A}} := \left[\mathbf{G}_{\mathbf{A}}^{(1)T},\ldots,\mathbf{G}_{\mathbf{A}}^{(I-1)T}\right]^T,\ \mathbf{G}_{\mathbf{A}}^{(m)} = \mathbf{D}_{\mathbf{A}}^{(m)}(C_2(\mathbf{A}^{(m)})*C_2(\mathbf{F}^{(m)})), \quad (15)$$

where the diagonal of $\mathbf{D}_{\mathbf{A}}^{(m)} = \text{Diag}\left(\mathbf{d}_{\mathbf{A}}^{(m)}\right)$ is constructed in accordance to (6).

c) Exploiting the shift-invariance structure of $\mathbf{B}$: Finally, using the shift-invariance property $\downarrow_n(\mathbf{B}) =\uparrow^n(\mathbf{B})\cdot\text{Diag}\left([y_1^n,\ldots,y_R^n]^T\right)$, an equation analogous to (12)

[1] Let us briefly provide intuition why the use of several shift factors m allows a better exploitation of the shift-invariance structure of an incomplete Vandermonde vector. As an example, consider the vector $\mathbf{w}*\mathbf{a} = [1\ x\ x^2\ 0\ x^4]^T$, where $\mathbf{w} = [1\ 1\ 1\ 0\ 1]^T$ is an indicator vector and $\mathbf{a} = [1\ x\ x^2\ -\ x^4]^T$ in which '−' denotes a missing entry. For $m = 1$, the shift-invariance of $\mathbf{a}$ yields

$$\downarrow_1(\mathbf{w}) * \downarrow_1(\mathbf{a}) = \uparrow^1(\mathbf{w}) * \uparrow^1(\mathbf{a})\cdot x, \quad (13)$$

in which $\downarrow_1(\mathbf{w}) = [1\ 1\ 0\ 1]^T$, $\downarrow_1(\mathbf{a}) = [x\ x^2\ -\ x^4]^T$, $\uparrow^1(\mathbf{w}) = [1\ 1\ 1\ 0]^T$ and $\uparrow^1(\mathbf{a}) = [1\ x\ x^2\ -]^T$. Since $\downarrow_1(\mathbf{w})*\uparrow^1(\mathbf{w}) = [1\ 1\ 0\ 0]^T$, it is clear that only the shift-invariance $[1\ x]^T\cdot x = [x\ x^2]^T$ of the subvector $[1\ x\ x^2]^T$ of $\mathbf{a}$ is exploited in (13), i.e., the entry x^4 has been ignored. Consider now the case where $m = 2$, i.e.,

$$\uparrow^2(\mathbf{w}) * \uparrow^2(\mathbf{a}) = \downarrow_2(\mathbf{w}) * \downarrow_2(\mathbf{a})\cdot x^2, \quad (14)$$

in which $\downarrow_2(\mathbf{w}) = [1\ 0\ 1]^T$, $\downarrow_2(\mathbf{a}) = [x^2\ -\ x^4]^T$, $\uparrow^2(\mathbf{w}) = [1\ 1\ 1]^T$ and $\uparrow^2(\mathbf{a}) = [1\ x\ x^2]^T$. Since $\downarrow_2(\mathbf{w})*\uparrow^2(\mathbf{w}) = [1\ 0\ 1]^T$, the shift-invariance $[1\ x^2]^T\cdot x^2 = [x^2\ x^4]^T$ of the subvector $[1\ x^2\ x^4]^T$ of $\mathbf{a}$ is exploited in (14), i.e., the entry x^4 plays a role as well. Overall, in the incomplete case we may consider all values of m for which we obtain a Vandermonde subvector of dimension at least three.

[2] Since $C_2\left(\mathbf{A}^{(m)}\right)$ is a row vector, '$\odot$' reduces to '$*$', i.e., $C_2\left(\mathbf{A}^{(m)}\right)\odot C_2\left(\mathbf{E}^{(m)}\right) = C_2\left(\mathbf{A}^{(m)}\right) * C_2\left(\mathbf{B}^{(m)}\right)$.

can be derived. Briefly, from (9) we can build the tensor

$$\mathcal{Z}^{(n)} = \mathcal{W}_{\mathbf{B}}^{(n)} * \Big(\sum_{r=1}^{R} \mathbf{b}_r^{(n)} \otimes \mathbf{f}_r^{(n)} \otimes \mathbf{s}_r\Big) \in \mathbb{C}^{2\times I(J-n)\times K}, \qquad (16)$$

with factor matrices $\mathbf{B}^{(n)} = \left[\begin{smallmatrix} 1 & \cdots & 1 \\ y_1^n & \cdots & y_R^n \end{smallmatrix}\right]$, $\mathbf{F}^{(n)} = \uparrow^n(\mathbf{B}) \odot \mathbf{A}$ and $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_R]$. The matrix $\mathbf{G}_{\mathbf{A}}$ in (15) is replaced by

$$\mathbf{G}_{\mathbf{B}} := \left[\mathbf{G}_{\mathbf{B}}^{(1)T}, \dots, \mathbf{G}_{\mathbf{B}}^{(J-1)T}\right]^T, \ \mathbf{G}_{\mathbf{B}}^{(n)} = \mathbf{D}_{\mathbf{B}}^{(n)}(C_2(\mathbf{B}^{(n)}) * C_2(\mathbf{F}^{(n)})). \qquad (17)$$

d) Combination of Khatri-Rao and shift-invariance structures: From (3), (12) and (16) it is clear that the incomplete URA factorization problem (3) can be translated into the coupled CPD problem ofthe set of tensors $\{\mathcal{Y}, \mathcal{Y}^{(m)}, \mathcal{Z}^{(n)}\}_{n=1,\dots,J-1}^{m=1,\dots,I-1}$ with missing fibers, which takes both the shift-invariance and Khatri-Rao structures into account. From (10), (15) and (17), we build

$$\mathbf{G} = \left[\mathbf{G}_{\mathbf{B}\odot\mathbf{A}}^T,\ \mathbf{G}_{\mathbf{A}}^T,\ \mathbf{G}_{\mathbf{B}}^T\right]^T. \qquad (18)$$

Proposition 3.1 now states that if

$$\begin{cases} \mathbf{S} \text{ in (3) has full column rank,} \\ \mathbf{G} \text{ in (18) has full column rank,} \end{cases} \qquad (19)$$

then $\mathbf{S}$ is essentially unique.

B. From coupled CPD to single-source DOA estimation problems

Assuming that $\mathbf{S}$ is essentially unique, the matrix $\mathbf{Z} = [\mathbf{z}_1, \dots, \mathbf{z}_R] = \mathbf{Y}(\mathbf{S}^T)^\dagger = \mathbf{D}_{\mathbf{w}}(\mathbf{B} \odot \mathbf{A})$ is also essentially unique. The remaining problem is to find the pair (x_r, y_r) in (2) from the vector $\mathbf{z}_r$, for $r \in \{1, \dots, R\}$.

We will first find y_r from the vector $\mathbf{z}_r = \mathbf{D}_{\mathbf{w}}(\mathbf{b}_r \otimes \mathbf{a}_r) = \mathrm{Diag}\left(\mathrm{Vec}\left(\mathbf{W}^T\right)\right)(\mathbf{b}_r \otimes \mathbf{a}_r) = \mathrm{Vec}\left(\mathbf{W}^T\right) * (\mathbf{b}_r \otimes \mathbf{a}_r)$, using the shift-invariance property $\uparrow^n(\mathbf{b}_r) \cdot y_r^n = \downarrow_n(\mathbf{b}_r)$. Denote $\hat{\mathbf{d}}_y^{(n)} = (\uparrow^n(\mathbf{I}_J) \otimes \mathbf{I}_I)\mathrm{Vec}\left(\mathbf{W}^T\right)$ and $\check{\mathbf{d}}_y^{(n)} = (\downarrow_n(\mathbf{I}_J) \otimes \mathbf{I}_I)\mathrm{Vec}\left(\mathbf{W}^T\right)$. Due to the shift-invariance property of $\mathbf{B}$, we obtain[3]

$$\mathbf{d}_y^{(n)} * (\uparrow^n(\mathbf{I}_J) \otimes \mathbf{I}_I)\mathbf{z}_r \cdot y_r^n = \mathbf{d}_y^{(n)} * (\downarrow_n(\mathbf{I}_J) \otimes \mathbf{I}_I)\mathbf{z}_r, \quad 1 \le n < J, \qquad (20)$$

where $\mathbf{d}_y^{(n)} = \hat{\mathbf{d}}_y^{(n)} * \check{\mathbf{d}}_y^{(n)}$ zeros the unobserved entries on either side of the equation. It can be verified that if

$$\omega(\mathbf{d}_y^{(1)}) > 0 \text{ or } \min(\omega(\mathbf{d}_y^{(i)}), \omega(\mathbf{d}_y^{(j)})) > 0 \qquad (21)$$

for some coprime pair (i, j), then y_r is unique. See [16] for details and for a polynomial rooting procedure that recovers y_r via (20). The generator x_r can be determined from $\mathbf{z}_r$ by switching the roles of x_r and y_r.

[3] We clarify relation (20) with an example. Consider the case where $I = 1$, $J = 5$, $\mathbf{a}_r = 1$ and $\mathbf{b}_r = [1\ -\ y_r^2\ y_r^3\ y_r^4]^T$ in which '−' denotes a missing entry. Hence, $\mathrm{Vec}\left(\mathbf{W}^T\right) = [1\ 0\ 1\ 1\ 1]^T$ and $\mathbf{z}_r = \mathrm{Vec}\left(\mathbf{W}^T\right)(\mathbf{b}_r \otimes \mathbf{a}_r) = [1\ 0\ 1\ 1\ 1]^T * (\mathbf{b}_r \otimes \mathbf{a}_r)$. For $n = 1$, the shift-invariance of $\mathbf{b}$ yields

$$\uparrow^1(\mathbf{b}_r)\otimes\mathbf{a}_r = [1\ -\ y_r^2\ y_r^3]^T,\ \hat{\mathbf{d}}_y^{(1)} = [1\ 0\ 1\ 1]^T,\ (\uparrow^1(\mathbf{I}_5) \otimes \mathbf{I}_1)\mathbf{z}_r = [1\ 0\ y_r^2\ y_r^3]^T,$$

$$\downarrow_1(\mathbf{b}_r)\otimes\mathbf{a}_r = [-\ y_r^2\ y_r^3\ y_r^4]^T,\ \check{\mathbf{d}}_y^{(1)} = [0\ 1\ 1\ 1]^T,\ (\downarrow_1(\mathbf{I}_5) \otimes \mathbf{I}_1)\mathbf{z}_r = [0\ y_r^2\ y_r^3\ y_r^4]^T.$$

It is clear that for $n = 1$ only the shift-invariance relation $[y_r^2\ y_r^3]^T \cdot y_r = [y_r^3\ y_r^4]^T$ can be exploited. This is formalized in the definition of $\mathbf{d}_y^{(1)} = \hat{\mathbf{d}}_y^{(1)} * \check{\mathbf{d}}_y^{(1)} = [0\ 0\ 1\ 1]^T$, i.e.,

$$[0\ 0\ y_r^2\ y_r^3]^T \cdot y_r = \mathbf{d}_y^{(1)} * (\uparrow^1(\mathbf{I}_5)\otimes\mathbf{I}_1)\mathbf{z}_r \cdot y_r = \mathbf{d}_y^{(1)} * (\downarrow_1(\mathbf{I}_5)\otimes\mathbf{I}_1)\mathbf{z}_r = [0\ 0\ y_r^3\ y_r^4]^T.$$

The Vandermonde structure of the subvector $[1\ y_r^2\ y_r^4]^T$ can be exploited by working with $n = 2$. The reasoning can be generalized to arbitrary I, J and m.

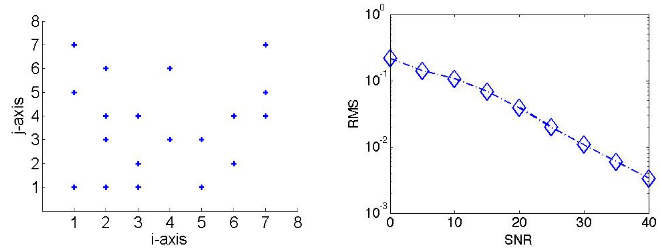

Fig. 1. (Left) Incomplete URA where '+' represents an antenna element. (Right) RMS error over 50 Monte Carlo runs, $R = 3$ and $K = 50$.

C. Summary and illustrative example

From the preceding discussion it follows that, if condition (19), (21) and its x-variant all are satisfied, then the generators $\{x_r, y_r\}$ of $\mathbf{A}$ and $\mathbf{B}$ are unique. Furthermore, they can be computed via the coupled CPD of the set of tensors $\{\mathcal{Y}, \mathcal{Y}^{(m)}, \mathcal{Z}^{(n)}\}_{n=1,\dots,J-1}^{m=1,\dots,I-1}$ with missing fibers followed by the rooting of a set of decoupled univariate polynomials.

Let us end the section with an illustrative example. Consider an incomplete $(I \times J)$ URA with $I = J = 7$ and where 19 out of the possible 49 sensor locations are used, as depicted in Figure 1 (Left). Consider the factorization $\mathbf{Y} = \mathbf{D}_{\mathbf{w}}(\mathbf{B} \odot \mathbf{A})\mathbf{S}^T$ in (9). The goal is to estimate the generators $\{x_r, y_r\}$ from $\mathbf{T} = \mathbf{Y} + \beta\mathbf{D}_{\mathbf{w}}\mathbf{N}$, where $\mathbf{N}$ is an unstructured perturbation matrix and $\beta \in \mathbb{R}$ controls the signal-to-noise ratio (SNR). In each trial of the Monte Carlo experiment, the generators $\{x_r, y_r\}$ are randomly drawn on the unit circle, and the real and imaginary entries of $\mathbf{S}$ and $\mathbf{N}$ are randomly drawn from a Gaussian distribution with zero mean and unit variance. The number of sources $R = 3$ and the number of snapshots used $K = 50$. The Root Mean Square (RMS) error over 50 Monte Carlo runs is shown in Figure 1 (Right). In the noiseless case the generators can be exactly recovered by GEVD and polynomial rooting up to $R \le \min(7, K)$.

V. Conclusion

CPD has already proven very useful in applications involving separable arrays. However, many interesting antenna configurations are not separable. In particular, in large-scale applications, sparse spatial sampling may be required. In this paper we showed that incomplete arrays enjoying shift-invariance and/or Khatri-Rao low-rank structures can be handled in the framework of coupled CPD with missing fibers.

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